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On the energy deficiency in self-preserving convective flows

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When similarity solutions are used to describe convective plumes or thermals, there is always found to be a discrepancy between the work done by buoyancy forces and the kinetic energy of mean motion. It is the main purpose of this paper to set down the ratio of these quantities for a wide variety of forms of buoyant elements and environmental stabilities. For consistency, the remaining fraction of the energy must appear as turbulent kinetic energy and eventually be dissipated, but these processes are not investigated in detail. The results are shown to have some relevance to the problem of convectively driven mixing across a density interface, where the largest scales of motion are dominant, and to the understanding of the transition zone between two self-preserving states of turbulent convection.

1. Introduction

Theories which predict the rate of mixing across stable density interfaces bounding convective layers require a knowledge of the kinetic energy possessed by the convective elements when they reach the interface. The energy input to the convection can be found from the buoyancy flux, but we also need to know the fraction of this energy which is dissipated by turbulence within the layer. Ball (1960) suggested that the dissipation will be negligible, and based a model of the mixing at an atmospheric inversion on this assumption, and Kraus & Turner (1967) used a related argument to discuss the seasonal behaviour of the oceanic thermocline.

The assumption of negligible dissipation can only be justified if the convective motions are of a scale comparable with the depth of the whole layer, but in a discussion of Ball's paper Scorer (1962) raised another possibility. If the dominant motion takes the form of 'thermals' rather than large cells, then the measurements of Woodward (1959) show that only about one third of the work done by buoyancy is present as kinetic energy of mean motion. Earlier measurements of Rouse, Yih & Humphreys (1952) show that there is a similar energy loss (to the turbulent components) in maintained buoyant plumes.

It is not necessary to rely only on the experiments quoted to find the ratio of kinetic energy of mean motion to the work done by buoyancy. It is shown below that this ratio can be calculated once the rate of variation of mean properties with height and the profile shapes have been specified, and results are obtained for similarity solutions corresponding to a buoyancy flux which varies as some power of the height. It turns out that the energy ratio is independent of the spreading angle and only moderately sensitive to the radial structure, but does depend more strongly on the assumed power laws, i.e. on the stability of the environment. The method used is related to the calculations of Priestley (1959, p. 88), though he concentrated on the heat flux produced by different types of convective element. The results for a uniform environment are also implied by Morton's (1971) discussion of the conservation equations for plumes.

2. Steady plumes

We consider first the case of steady turbulent plumes rising in a calm atmosphere from point or line sources. We shall assume that the velocity and density distributions are similar at all heights and that the scale widths for velocity and buoyancy are identical. (The latter condition can easily be relaxed.) A class of similarity solutions may be defined, these being characterized by a linear increase of radius with height z and a vertical velocity proportional to an arbitrary power of z, which is measured from the source level (Batchelor 1954). The compatible variation of the density difference between the plume and its environment can be calculated; this may be thought of as being caused by an environmental density variation, or by a height-dependent internal mechanism such as condensation (Turner 1969).

It is convenient here to derive the desired relations between the mean energy flux and work done by buoyancy from first principles, rather than relating them by a hypothetical power-law density distribution in the environment. Suppose that the plume is axisymmetric and let the vertical velocity and density fields be described by

$$g(\rho_0-\rho)/\rho_1 = \Delta = Az^p f(\eta), \tag{1}$$

$$w = Bz^q f(\eta), \tag{2}$$

where ρ_0 is the environmental density, ρ is that inside the plume and ρ_1 is a reference density. Here $\eta = (r/z)^2$ and $f(\eta)$ is a radially symmetric function which will be left arbitrary; a linear spread is implied in writing down these forms, but the angle of spread does not enter explicitly.

The flux E of mean kinetic energy across a horizontal plane at height z and the rate W of working of the buoyancy forces are then given by

$$E = \pi\rho \int_0^\infty w^3 r\, dr = \tfrac{1}{2}\pi\rho B^3 z^{3q+2} \int_0^\infty f^3\, d\eta \tag{3}$$

and

$$W = 2\pi\rho \int_0^\infty w\Delta r\, dr = \pi AB\rho z^{p+q+2} \int_0^\infty f^2\, d\eta. \tag{4}$$

Form of profile	Value of q	Interpretation of special case	Energy ratio
Gaussian	$-\frac{1}{3}$	Neutral environment	0·500
	0	Constant velocity	0·667
	$\frac{1}{3}$	Free convection	0·750
	$\frac{1}{2}$	Constant density difference	0·778
	1	Saturated environment	0·834
'Top hat'	$-\frac{1}{3}$	Neutral environment	0·375
	0	Constant velocity	0·500
	$\frac{1}{3}$	Free convection	0·562
	$\frac{1}{2}$	Constant density difference	0·583
	1	Saturated environment	0·625
Measured, Rouse *et al.*		Neutral environment	0·57

TABLE 1. The fraction of the work done by buoyancy forces which is accounted for in the kinetic energy of mean motion according to (8). Results are given for axisymmetric plumes with various profiles and power-law variations of velocity with height, $w \propto z^q$.

Relations between A and B, and p and q can be obtained using the momentum equation integrated across the plume, i.e.

$$\frac{\partial}{\partial z}\left[2\pi\int_0^\infty w^2 r\,dr\right] = 2\pi\int_0^\infty \Delta r\,dr$$

or

$$\frac{\partial}{\partial z}[B^2 z^{2q+2}]\int_0^\infty f^2 d\eta = Az^{p+2}\int_0^\infty f d\eta. \tag{5}$$

Thus

$$p = 2q-1 \tag{6}$$

and

$$2(q+1)B^2\int_0^\infty f^2 d\eta = A\int_0^\infty f d\eta. \tag{7}$$

We can now form the ratio of the divergence dE/dz of the mean flow energy flux to the rate of working of the buoyancy forces: the fraction of the work done which is accounted for in this way is

$$\frac{dE/dz}{W} = \frac{3q+2}{4(q+1)}I_1, \tag{8}$$

where

$$I_1 = \frac{\int_0^\infty f d\eta \int_0^\infty f^3 d\eta}{[\int f^2 d\eta]^2}. \tag{9}$$

The energy ratio depends both on the exponent q (which is determined by the stability of the environment, or the nature of the buoyancy-producing internal mechanism) and on I_1, a profile constant.

The ratios corresponding to special values of q and I_1 are summarized in table 1. Two widely used theoretical profiles have been compared, the 'top hat' with $I_1 = 1$, and the Gaussian form, for which $I_1 = \frac{4}{3}$. Some of the values chosen for q require explanation. The appropriate value in neutral surroundings is $q = -\frac{1}{3}$, since this and the corresponding $p = -\frac{5}{3}$ lead to a constant flux of buoyancy with height in the plume $\left(\text{i.e. } \int_0^\infty \Delta wr\,dr \text{ is independent of height}\right)$. Constancy of other

Form of profile	Value of q	Interpretation of special case	Energy ratio
Gaussian	0	Neutral environment	0·578
	$\frac{1}{3}$	Free convection	0·695
	$\frac{1}{2}$	Constant density difference	0·723
	1	Saturated environment	0·771
'Top hat'	0	Neutral environment	0·500
	$\frac{1}{3}$	Free convection	0·600
	$\frac{1}{2}$	Constant density difference	0·625
	1	Saturated environment	0·667
Measured, Rouse *et al.*		Neutral environment	0·53

TABLE 2. The fraction of the work done by buoyancy forces which is accounted for in the kinetic energy of mean motion according to (10). Line plumes; various profiles and power-law variations of velocity with height, $w \propto z^q$.

parameters can also be achieved by appropriate choice of q. The 'free convection' regime is taken, following Priestley (1959), to be that corresponding to Δw being independent of height, and $q = 1$ corresponds to the case of a rising plume of moist air in a saturated environment (Ludlam 1958).

Also shown in table 1 is the ratio derived from their measurements by Rouse *et al.* (1952). Their calculations properly took into account the differences in the width of the velocity and temperature profiles, but used Gaussian profiles fitted to the data. Since this value results from a single set of laboratory experiments it is subject to experimental error, unlike the other values in the table which are direct deductions from the assumptions.

Entirely analogous results can be obtained for line plumes by integrating over a section normal to the plume axis; only the results will be given here. With, again, $w \propto z^q$, (8) is replaced by

$$\frac{dE/dz}{W} = \frac{3q+1}{2(2q+1)} I_2, \tag{10}$$

where

$$I_2 = \frac{\int_{-\infty}^{\infty} f\,d\zeta \int_{-\infty}^{\infty} f^3\,d\zeta}{\left[\int_{-\infty}^{\infty} f^2\,d\zeta\right]^2}, \tag{11}$$

with $f = f(\zeta)$, $\zeta = x/z$ (x being the horizontal distance from the plume axis). The profile factor I_2 is unity for the 'top hat' profiles and $2/\sqrt{3}$ for the Gaussian form. The ratios resulting from particular choices of q are shown in table 2. The values of q corresponding to a given physical situation can of course be different from those for axially symmetric plumes. For the line plume, a neutral environment and both a constant velocity and constant heat flux are implied by $q = 0$.

3. Suddenly released point sources

3.1. *Distributed vorticity*

A parallel argument to the above can be used to discuss turbulent 'thermals' which are assumed to retain similar velocity and density distributions as they rise and mix with their environment. Dimensional arguments again show that

Type of element	Value of q	Interpretation of special case	Energy ratio $(\partial T/\partial z)/F$
Hills spherical vortex	-1	Neutral environment	0·357
	0	Constant velocity	0·714
	$\frac{1}{3}$	Free convection	0·786
	$\frac{1}{2}$	Constant density difference	0·817
	1	Saturated environment	0·893
Similarity solution, small core	$R/a = 5$	Neutral environment	0·564
	$R/a = 20$	Neutral environment	0·689
Constant core volume		Neutral environment	1·00

TABLE 3. The fraction of the work done by buoyancy forces which is accounted for in the kinetic energy of mean motion for suddenly released buoyant elements of various kinds.

the radius will increase linearly with height for the class of buoyant elements whose mean density and velocity are described by

$$\bar{\Delta} = Cz^p, \quad \bar{w} = Dz^q. \tag{12}$$

The magnitudes of C and D will depend on the specific profiles assumed.

It has been shown by Levine (1959) and Turner (1964) that the velocity distribution known as Hill's spherical vortex gives a fairly good instantaneous representation of laboratory thermals, so results will first be developed for this case. In Hill's vortex, all the vorticity is contained in a sphere of radius r, with potential flow outside it.

The impulse P and total kinetic energy T are given in terms of r by

$$P = 2\pi\rho r^3\bar{w}, \quad T = \tfrac{10}{7}\pi\rho r^3\bar{w}^2. \tag{13}$$

Using the fact that $r \propto z$ and substituting for $\bar{w}$ from (12) gives

$$\frac{\partial T}{\partial z} = \tfrac{10}{7}\pi\rho D^2 r^3(2q+3)z^{2q-1}. \tag{14}$$

This is to be compared with the total buoyancy force F, which from the momentum equation is just $\partial P/\partial t$. Thus

$$F = \frac{\partial P}{\partial t} = \frac{\partial P}{\partial z}.\bar{w}$$

or

$$\tfrac{4}{3}\pi\rho r^3\bar{\Delta} = 2\pi\rho D^2 r^3(q+3)z^{2q-1}. \tag{15}$$

Comparing with (12), we obtain $p = 2q-1$ as before and also a relation between C, D and q:

$$C = \tfrac{3}{2}(q+3)D^2. \tag{16}$$

Without knowing anything about the density distribution (and indeed without using (16)) the right-hand side of (15) can be compared with (14) to give

$$\frac{\partial T/\partial z}{F} = \frac{10}{7}\frac{q+\frac{3}{2}}{q+3}. \tag{17}$$

Again the angle of spread does not enter into the final result. The values of this ratio appropriate to different environments are shown in table 3. The requirement in neutral surroundings is now that F must remain constant, which using (15) (and remembering that $r \propto z$) leads to $q = -1$.

For Hill's spherical vortex the kinetic energy of mean motion may be split up still further. The factor $\frac{10}{7}$ in (13) and (17) can be regarded as made up of three parts: $\frac{1}{3}$ is due to the kinetic energy of external fluid flowing round the spherical region, $\frac{2}{3}$ to the sphere moving as a solid body and the balance of $\frac{3}{7}$ is due to the kinetic energy of the internal circulation. Thus, for example, in a neutral environment only $\frac{2}{3} \times \frac{1}{4} = \frac{1}{6}$ of the work done by buoyancy appears as kinetic energy of the spherical region moving as a solid.

3.2. *Concentrated vortex cores*

Another possible form of convecting element is one in which the buoyancy and vorticity are not distributed throughout the moving volume, but are contained in a much sharper (possibly turbulent) toroidal core which carries an enclosed region of irrotational fluid with it. When the radius a of the cross-section of the core is small compared to that (R) of the toroid, expressions for the impulse, velocity and kinetic energy may be written down in terms of the circulation K. According to Lamb (1932, p. 241) these are respectively

$$\left.\begin{aligned} P &= \pi\rho K R^2, \\ \overline{w} &= (K/4\pi R)\{\ln(8R/a) - \tfrac{1}{4}\}, \\ T &= \tfrac{1}{2}\rho K^2 R\{\ln(8R/a) - \tfrac{7}{4}\}. \end{aligned}\right\} \tag{18}$$

In this section we shall consider only buoyant elements with constant total buoyancy in a neutrally stable environment (though the argument leading to (17) can readily be generalized). In this case the circulation K also remains constant as R increases linearly with distance z above the (virtual) point source. First, let us suppose that the distribution of vorticity remains simiar at all heights as the ring expands; Turner (1957) reported laboratory experiments of this kind in which turbulent cores increased their radius at a rate proportional to that of the ring. Even if the core size is too great for (18) to hold exactly, dimensional arguments show that similar relations must apply, with constants C_1 and C_2 replacing the terms in curly brackets in the equations for velocity and kinetic energy.

The argument used in deriving (17) may be applied again to give

$$\frac{\partial T}{\partial z} = \frac{C_2}{C_1}\frac{\partial P}{\partial t} = \frac{C_2}{C_1}F. \tag{19}$$

A spherical vortex may be treated as a special case of (19), for which it can be shown that $C_1 = \frac{4}{5}\pi(\frac{2}{5})^{\frac{1}{2}}$ and $C_2 = \frac{2}{7}\pi(\frac{2}{5})^{\frac{1}{2}}$, so that $C_2/C_1 = \frac{5}{14} = 0{\cdot}357$ as before. For the distributions (18), $C_1 = \ln(8R/a) - \frac{1}{4} = C_2 + \frac{3}{2}$, and the energy ratios calculated for two core sizes (on the assumption that (18) still hold for these ratios of R/a) are shown in table 3.

The result just obtained shows that an increasing fraction of the work done by buoyancy appears as kinetic energy of mean motion as the vorticity becomes more and more concentrated. Unless the core becomes extremely small, however, it is impossible to account for *all* the work done, while still retaining the similarity assumption. If this assumption is relaxed, it *is* possible to obtain an energy balance in the following way.

Let us now suppose that the buoyancy and vorticity are contained in a core which is spreading out so slowly compared with the rate of change of R that its volume can be regarded as substantially constant. (It is not always possible to neglect the diffusion, especially of vorticity, in this way but the consequences of this extreme assumption are worth pursuing.) Thus a^2R = constant, and an increase in the radius R is compensated by a decrease in the core cross-section. Substituting for a in the expression for kinetic energy (18), and treating R as a function of time wherever it occurs, it follows on differentiation that

$$\begin{aligned}\frac{\partial T}{\partial t} &= \tfrac{1}{2}K^2\rho\frac{\partial R}{\partial t}\{\ln(8R/a)-\tfrac{1}{4}\} \\ &= 2\pi K\rho R\frac{\partial R}{\partial t}\overline{w} = \frac{\partial P}{\partial t}\overline{w}. \end{aligned} \tag{20}$$

Thus the variation of R/a in this particular way allows an exact balance to be achieved between the work done by buoyancy and the mean kinetic energy produced.

Several pieces of experimental evidence have suggested that vortex rings approximating to this form can exist in practice. In the experiments of Turner (1957), mentioned above, laminar buoyant cores were sometimes produced which for a time became thinner as R increased (though they later broke down to the turbulent 'similarity' state on which most of the measurements were made). A more spectacular example was observed in the atmosphere by E. G. Bowen, and described in a previously reported letter to Sir Geoffrey Taylor (see Turner 1957). A vortex ring formed from the smoke of a small explosion on the ground rose to a height of about five thousand feet, with the cross-section of the visible core decreasing and remaining extremely sharp until the ring broke up. This final breakup was attributed by the observers to the occurrence of condensation within the core, though it now seems more likely to be a consequence of the increasing viscous dissipation near the rapidly rotating core at later times, which makes it no longer consistent to retain the profiles appropriate to zero energy discrepancy. A detailed discussion of this effect is outside the scope of the present paper.

4. Discussion

With a range of special results now at hand, we can consider some of the more general implications. First, it is clear that the efficiency with which convective motions convert potential to kinetic energy on the largest scale does depend on the form of the motion, and on the velocity profile. In neutrally stratified surroundings, between about $\frac{1}{3}$ and $\frac{3}{5}$ of the energy is accounted for in this way, the smallest fraction being obtained for an isolated spherical vortex and the largest for a line plume and Gaussian profiles. In one special case, a non-similar vortex with constant core volume, all the energy is accounted for when the motion is assumed to be inviscid and non-diffusive; but as indicated above, a vortex of this kind in a real fluid will eventually break down and take on a different form for which there is again a loss of energy to turbulence.

If it is the largest scale motions which are responsible for mixing near an interface bounding a convecting region (and this seems likely from direct observation), then the rate of mixing will depend on the form of the convection elements, not just on the convective heat flux. Line plumes should be most effective, and we should refer again to the evidence presented by Priestley (1959) which suggests that convection in the lower atmosphere does characteristically take the form of plumes aligned downwind. On the basis of the energy ratios calculated here, 'thermals' should be least effective, particularly when we remember that the kinetic energy estimates include the environmental motion as well as the forward motion and the internal circulation. The relative importance of the last two contributions is not yet clear.

When the convection takes place under conditions of decreasing stability (i.e. as q increases), a larger fraction of the work done by buoyancy appears as mean kinetic energy. The difference between the neutral and most unstable case treated (corresponding to a saturated environment in cloudy convection) is most dramatic for the spherical vortex, but it is still large for the axially symmetric maintained plume. There are several pieces of observational evidence which bear on this point and support the general form of the predicted dependence on q. The peak vertical velocity varies surprisingly little between small and large clouds (Warner 1970), but since mean updraughts are greater in the latter case, the relative turbulence level must be lower. Glider pilots also report that the cores of vigorous convective clouds are relatively smooth, suggesting that a low level of turbulence is associated with the strong accelerations in such clouds.

This brings us to a central (and unanswered) question underlying this study: what exactly has happened to the energy deficit implied by these calculations? All we have said so far is that this is not present in mean motion and must therefore have been dissipated viscously or converted into turbulent kinetic energy. It will be shown elsewhere that an energy argument can also be used to put an upper limit on the scale of a laminar flow in which all the energy can be dissipated viscously, but little can be deduced using this method alone about the level of turbulence or, equivalently, the rate of dissipation at the smallest scales of motion. The results suggest, however, that there will be a different relation between the mean flow and turbulent kinetic energies for each similarity condition (i.e. each value of q). This difference can be important in the assessment of various 'entrainment' models of mixing into the convective elements. The first uses of this idea (e.g. Morton, Taylor & Turner 1956; Squires & Turner 1962) were based on the assumption of an inflow velocity proportional to the mean upward velocity; the ratio between these was taken as fixed and independent of stability. Telford (1966) and Morton (1968) have suggested instead that the entrainment rate should be based on the level of turbulence or on the Reynolds stress. These assumptions will not only change the 'entrainment constant' from one value to another as q is changed, but they will also build a 'memory' into the system; calculations of the mixing into a flow which is in transition between two similarity states can no longer be based only on the local properties but must take account of the history.

The present results clearly cannot provide detailed answers to questions such as these, but they may give some guidance as to the likely errors involved in applying the simplest entrainment models to arbitrary flows. If the whole history of a convection element can be described by power-law variations within a limited range of q, then the unassigned fraction of the energy will change little, and the entrainment rate can plausibly be related to the mean flow. When there is a large change in the behaviour, however, say a strong acceleration followed by deceleration, the energy ratios vary so much that this simple model must be abandoned. The assumption that mixing is related to the mean velocity is especially poor when a convection element is coming to rest and reversing its motion, since at that time vigorous turbulent mixing can still be going on. The energy available to produce turbulence in this latter case can, incidentally, be estimated using energy arguments related to those developed in §§2 and 3. These will not be reproduced here, since the assumption that similarity of profiles is maintained during the deceleration is more difficult to justify.

This study grew out of a series of stimulating conversations with Dr C. G. H. Rooth, now of the School of Marine and Atmospheric Sciences of the University of Miami, while both of us were working at the Woods Hole Oceanographic Institution some years ago. My interest in it was revived during a visit to Miami on the way to spend a period of leave in the Cloud Physics group of C.S.I.R.O., Australia, and it was continued and extended during that time. I am grateful to Mr J. Warner for his support and encouragement, and to Prof. B. R. Morton for many illuminating discussions on the subject of plumes and vortices.

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